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UNIVERSALLY MEASURE CONTINUUM-WISE EXPANSIVE HOMOCLINIC CLASSES

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Abstract. Investigating local dynamics requires precise control to effectively manage the subtle differences that distinguish it from global dynamics. This paper aims to study the localized perspective of the recently proposed continuum-wise expansive measures [13]. Let $f : M \to M$ be a diffeomorphism on a closed smooth manifold M and let p be a hyperbolic periodic point of f. We prove that if the homoclinic class $H_f(p)$ of f associated to p is C^1-robustly measure continuum-wise expansive then it is hyperbolic.

1. Introduction

Let M be a closed connected smooth Riemannian manifold without boundary. Denote by Diff(M) be the set of diffeomorphisms $f : M \to M$ with the C^1 topology. Let d be the distance on M induced from a Riemannian metric $\| \cdot \|$ on the tangent bundle TM. A diffeomorphism f is said to be *expansive* if there is $e > 0$ such that for any $x, y \in M$ if $d(f^i(x), f^i(y)) < e$ for all $i \in \mathbb{Z}$ then $x = y$. Recently, there has been a lot of research on various expansive (N-expansive [9], countably expansive [11], continuum-wise expansive [5], etc) systems of dynamical systems. They play a key role in the study of qualitative theories of dynamical systems such as stability.

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In this study, we consider a general type of measure expansive systems which is called measure continuum-wise expansive system [13]. Continuum-wise expansiveness is stronger than pointwise expansiveness, which makes it a more challenging property to study, but also a more powerful one for characterizing chaotic behavior [5]. The continuum concept has recently been spotlighted as one of the interesting research subjects by many researchers in dynamical systems [1, 2, 10].

We introduce the essential concepts of continuum-wise expansive systems. A continuum-wise expansive system is a type of dynamical system that the system is sensitive to small perturbations across a set of points in the underlying space, called the continuum. By a continuum, we mean a compact metric connected nondegenerate space. A *subcontinuum* is a nonempty subset which is a continuum with respect to the induced topology. We say that it is *degenerated* if it reduces to a single point. A diffeomorphism f is said to be *continuum-wise expansive* (*cw-expansive* for short) if there is a constant $e > 0$ such that for any nondegenerate continuum A there is an integer $n \in \mathbb{Z}$ such that $\text{diam} f^n(A) \geq e$, where $\text{diam} A = \sup\{d(x, y) : x, y \in A\}$ for any subset A of M. Here the constant e is called an *cw-expansive constant* for f. It is clear that if a diffeomorphism is expansive then it is cw-expansive, but the converse is not true. It is well known that $\mathcal{S}^2$ does not admit an expansive diffeomorphism, but it admits a cw-expansive diffeomorphism [3]

A homoclinic class is a set of points in phase space that share the same unstable and stable manifolds of a common hyperbolic equilibrium or periodic orbit. Homoclinic classes can exhibit complex and interesting dynamical behaviors in diffeomorphisms, such as the existence of horseshoe maps and other types of chaos. A point $x \in M$ is called a *periodic point* if there is $\pi(x) > 0$ such that $f^{\pi(x)}(x) = x$, where $\pi(x)$ is the period of x. A periodic point p with period $\pi(p) > 0$ is considered *hyperbolic* if the derivative $D_p f^{\pi(p)}$ has no eighenvalues with norm one. Let $Per(f) = \{x \in M : x \text{ is a periodic point of} f\}$. Let $p, q \in Per(f)$ be hyperbolic. We say that p and q are *homoclinically related* if $W^s(p) \pitchfork W^u(q) \neq \emptyset$ and $W^u(p) \pitchfork W^s(q) \neq \emptyset$, and in such a case, we write $p \sim q$. Let us denote $H_f(p) = \overline{\{q \in Per(f) : p \sim q\}}$. It is known that $H_f(p)$ is a closed, f-invariant, and transitive set.

We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a continuous Df-invarinat splitting $E^s \oplus E^u$ and there exist constants $C > 0, 0 < \lambda < 1$ such that

$$\|Df^n|_{E^s(x)}\| \leq C\lambda^n \quad \text{and} \quad \|Df^{-n}|_{E^u(x)}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. It is known that if Λ is hyperbolic for f then f is expansive. If $\Lambda = M$ then we say that f is *Anosov*.

The main purpose of this paper is to characterize homoclinic classes $H_f(p)$ containing a hyperbolic periodic point p by making use of a general type of continuum-wise expansiveness under C^1 open condition. This is a generalization of the main result in [6].

Main Theorem. Let p be a hyperbolic periodic point of $f \in \mathrm{Diff}(M)$. If the homoclinic class $H_f(p)$ is C^1-robustly measure continuum-wise expansive, then it is hyperbolic.

2. Measure Continuum-wise Expansiveness

Measure theory provides a rigorous mathematical framework for studying the behavior of differentiable dynamical systems, and it allows us to make precise statements about the long-time behavior of these systems. In a similar vein, Shin study Kato's continuum-wise expansivity for measure view points through the notion of continuum-wise expansive measure [13] which suggest a link between the continuum theory and measurable dynamics. We introduce the extended cw-expansiveness to measures through to the notion of cw-expansive measure.

Definition 2.1. [13] Let μ be a Borel probability measure which is not necessarily f-invariant. We say that f is measure continuum-wise expansive (or simply, measure cw-expansive, μ-cw-expansive) if there is $c > 0$ such that for every subcontinuum A of M with $\mu(A) > 0$ there is $n \in \mathbb{Z}$ such that $diam(f^n(A)) > c$.

Note that if a diffeomorphism $f : M \to M$ is μ-cw-expansive for $\mu \in \mathcal{M}(M)$, then μ is clearly nonatomic. This means that μ-cw-expansive system is generalized of cw-expansive system. This property holds almost everywhere.

One of the topics recently actively studied is to understand how robust dynamics properties on maniolds characterize dynamics properties on tangent bundles. If there is no further mention, we follow the measure defined in Definition 2.1. We say that f is *C^1-robustly measure cw-expansive* if there is a C^1-neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}(M)$ of f such that for any $g \in \mathcal{U}(f)$, g is measure cw-expansive.

With these motivations we introduce a general measure cw-expansive concept in this work. Let $\mathcal{M}(M)$ be the set of all Borel probability measures on M endowed with the weak* topology, and let $\mathcal{M}^*(M)$ be the set of nonatomic measures $\mu \in \mathcal{M}(M)$.

DEFINITION 2.2. For any $\mu \in \mathcal{M}^*(M)$, we say that $f \in \mathrm{Diff}(M)$ is universally measure continuum-wise expansive (or simply, universally measure cw-expansive, universally μ-cw-expansive) if there is $c > 0$ such that for every subcontinuum A of M, if $diam(f^n(A)) \leq c$ for all $n \in \mathbb{Z}$ then $\mu(A) = 0$.

In the absence of explicit mention, we consider measure cw-expansive ness shall be regarded as universally measure cw-expansiveness throughout this paper.

We want to extend the locality by proving that the dynamical properties of the whole system hold true for local subsystems. To establish this localization, we introduce the local version as follows.

DEFINITION 2.3. We say that $f \in \mathrm{Diff}(M)$ has the C^1-robustly measure cw-expansive on a subset Λ of M if a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ exist such that

(i) $\Lambda = \cap_{n \in \mathbb{Z}} f^n(U)$, and
(ii) for any $g \in \mathcal{U}(f)$, g has the measure cw-expansive property on the continuation Λ_g of Λ.

3. Proof of Main Theorem

To prove the main theorem, we have the same philosophy of Sambarino and Vietz in [12]. We need several lemmas for completing the proof. We recall the concept of a local star condition. For any closed f-invariant set $\Lambda \subset M$, we say that a diffeomorphism f is a *star* on Λ if a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ exist such that for any $g \in \mathcal{U}(f)$, every $p \in \Lambda_g \cap P(g)$ is hyperbolic, where $\Lambda_g = \Lambda_g(U) = \bigcap_{n \in \mathbb{Z}} g^n(U)$ is the continuation of Λ. We denote by $\mathcal{F}(\Lambda)$ the set of all diffeomorphisms that are stars on Λ.

LEMMA 3.1. *Let Λ be a closed invariant set of f. If f is C^1-robustly measure cw-expansive on Λ, then $f \in \mathcal{F}(\Lambda)$.*

Proof. Suppose that f exhibits the C^1 robustly measure cw-expansive property on Λ. By the definition of $\mathcal{F}(\Lambda)$, a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ exist such that for any $g \in \mathcal{U}(f)$, every $p \in \Lambda_g \cap P(g)$ is hyperbolic.

By contradiction, we assume that $f \notin \mathcal{F}(\Lambda)$. Since f is C^1-robustly measure cw-expansive, there exists a C^1-neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}(M)$

of f such that for any $g \in \mathcal{U}(f)$ and any $\mu \in \mathcal{M}_g^*(M)$, g is μ-cw-expansive. Since $f \notin \mathcal{F}(\Lambda)$, we can take $g \in \mathcal{U}(f)$ and non-hyperbolic periodic point p of g.

By Franks' Lemma [4], with a small modification of the map g with respect to the C^1-topology, we may assume that $D_p g^{\pi(p)}$ has only one eigenvalue λ with modulus equal to 1 (or only one pair complex conjugated eigenvalues). Denote by E_p^c the eigenspace corresponding to λ.

Case 1 : $dim E_p^c = 1$

Suppose that $\lambda = 1$ for simplicity. Then we have $\varepsilon_0 > 0$ and $\phi \in \mathcal{U}(f)$ such that

$$\phi^{\pi(p)}(p) = g^{\pi(p)}(p) = p$$

and

$$\phi(x) = exp_{g^{i+1}(p)} \circ D_{g^i(p)} g \circ exp^{-1}_{g^i(p)}(x)$$

if $x \in B_{\varepsilon_0}(g^i(p))$ for $0 \leq i \leq \pi(p) - 2$, and

$$\phi(x) = exp_p \circ D_{g^{\pi(p)-1}(p)} g \circ exp^{-1}_{g^{\pi(p)-1}(p)}(x)$$

if $x \in B_{\varepsilon_0}(g^{\pi(p)-1}(p))$.

Since the eigenvalue λ of $D_p g^{\pi(p)}|_{E_p^c}$ is 1, there is a small arc

$$\mathcal{L}_p \subset B_{\varepsilon_0}(p) \cap exp_p(E_p^c(\varepsilon_0))$$

with its center at p such that

- $\phi^i(\mathcal{L}_p) \cap \phi^j(\mathcal{L}_p) = \emptyset$ for $0 \leq i \neq j \leq \pi(p) - 1$,
- $\phi^{\pi(p)}(\mathcal{L}_p) = \mathcal{L}_p$, and
- $\phi^{\pi(p)}|_{\mathcal{L}_p}$ is the identity map.

Here $E_p^c(\varepsilon_0)$ is the ε_0-ball in E_p^c centered at the origin $\mathcal{O}(p)$.

Let $m_{\mathcal{L}_p}$ be a normalized Lebesgue measure on $\mathcal{L}_p$. We define $\mu \in \mathcal{M}_\phi(M)$ by

$$\mu(C) = \frac{1}{\pi(p)} \sum_{j=0}^{\pi(p)-1} m_{\mathcal{L}_p}(\phi^{-j}(C \cap \phi^j(\mathcal{L}_p)))$$

for any Borel set C of M. We can take an expansive constant $\delta = \varepsilon_0 > 0$. Since $\phi^{\pi(p)}|_{\mathcal{L}_p}$ is the identity map,

$$\{y \in \mathcal{L}_p : d(p, y) < \delta_1\} \subset \Gamma_\delta(p).$$

Thus, we have

$$\mu(\Gamma_\delta(p)) \geq \mu(\{y \in \mathcal{L}_p : d(p, y) < \delta_1\}) > 0,$$

which implies ϕ is not μ-cw-expansive. But this contradicts with $\phi \in \mathcal{U}(f)$.

Case 2 : $dimE^c_p = 2$

To avoid notational complexity, we consider only the case $g(p) = p$. Then we have $\varepsilon_0 > 0$ and $\phi \in \mathcal{U}(f)$ such that

$$\phi(p) = g(p) = p$$

and

$$\phi(x) = exp_{g(p)} \circ D_p g \circ exp_p^{-1}(x)$$

if $x \in B_{\varepsilon_0}(p)$. With a small modification of the map $D_p g$, we may suppose that there is $l > 0$ such that $D_p g^l(v) = v$ for any $v \in E^c_p(\varepsilon_0)$.

Take $v_0 \in E^c_p(\varepsilon_0)$ such that $\|v_0\| = \varepsilon_0/4$, and set

$$\mathcal{J}_p = exp_p(\{t \cdot v_0 : 1 \le t \le 1 + \varepsilon_0/4\}).$$

Then $\mathcal{J}_p$ is an arc such that

· $\phi^i(\mathcal{J}_p) \cap \phi^j(\mathcal{J}_p) = \emptyset$ for $0 \le i \ne j \le l-1$,
· $\phi^l(\mathcal{J}_p) = \mathcal{J}_p$, and
· $\phi^l|_{\mathcal{J}_p}$ is the identity map.

Let $m_{\mathcal{J}_p}$ be the normalized Lebesgue measure on $\mathcal{J}_p$ and set

$$\mu(C) = \frac{1}{l} \sum_{j=0}^{l-1} m_{\mathcal{J}_p}(\phi^{-j}(C \cap \phi^j(\mathcal{J}_p)))$$

for a Borel set C. Then $\mu \in \mathcal{M}^*_\phi(M)$. Thus ϕ is not μ-cw-expansive which contradicts $\phi \in \mathcal{U}(f)$. □

By Proposition II.1 in [7] and the above lemma, we get the following lemma.

LEMMA 3.2. *Suppose that the homoclinic class $H_f(p)$ is C^1-robustly measure cw-expansive, and le $\mathcal{U}_0(f)$ as the Lemma 3.1. Then there are constants $C > 0, 0 < \lambda < 1$ and $m > 0$ such that*

- *for any $g \in \mathcal{U}_0(f)$, if $q \in \Lambda_g \cap P(g)$ has minimum period $\pi(q) \ge m$, then*

$$\prod_{i=0}^{k-1} \|D_{g^{im}(q)} g^m|_{E^s_{g^{im}(q)}}\| < C\lambda^k \quad \text{and} \quad \prod_{i=0}^{k-1} \|D_{g^{-im}(q)} g^m|_{E^u_{g^{-im}(q)}}\| < C\lambda^k$$

  where $k = [\pi(q)/m]$.
- *$H_f(p)$ admits a dominated splitting $T_{H_f(p)}M = E \oplus F$ with* $\dim E =$ index(p).

If an invariant set Λ admits a dominated splitting, then Mañé has shown the existence of locally invariant manifolds everywhere on Λ which are tangent to the invariant subspaces of the splitting [8]. From [8], the set $W^{cs}_\varepsilon(x)$ and $W^{cu}_\varepsilon(x)$ are called the local center stable and local center unstable manifolds of x, respectively. The following lemma can be proved similarly to that of Lemma 4 in [12].

LEMMA 3.3. *Let $H_f(p)$ be the homoclinic class of f associated to a hyperbolic periodic point p, and suppose that $H_f(p)$ is C^1-robustly measure cw-expansive. Then for C, λ as in Lemma 3.2 and $\delta > 0$ satisfying $\lambda' = \lambda(1+\delta) < 1$ and $q \sim p$, there exists $0 < \varepsilon_1 < \varepsilon$ such that if for all $0 \leq n \leq \pi(q)$ it holds that for some $\varepsilon_2 > 0$, $f^n(W^{cs}_{\varepsilon_2}(q)) \subset W^{cs}_{\varepsilon_1}(f^n(q))$ then $f^{\pi(q)}(W^{cu}_{\varepsilon_2}(q)) \subset W^{cs}_{C\lambda'^{\pi(q)}\varepsilon_2}(q)$. Similarly, if $f^{-n}(W^{cu}_{\varepsilon_2}(q)) \subset W^{cu}_{\varepsilon_1}(f^{-n}(q))$ then $f^{-\pi(q)}(W^{cu}_{\varepsilon_2}(q)) \subset W^{cu}_{C\lambda'^{\pi(q)}\varepsilon_2}(q)$.*

If $H_f(p)$ is not hyperbolic then it may contain periodic points having different indices. Recall that a compact f-invariant set Λ has a *local product structure* if given $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d(x, y) < \delta$ and $x, y \in \Lambda$ then

$$\emptyset \neq W^s_\varepsilon(x) \cap W^u_\varepsilon(y) \subset \Lambda.$$

LEMMA 3.4. *Let $H_f(p)$ be C^1-robustly measure cw-expansive. Then $H_f(p)$ has a local product structure. Moreover, for any $q \in H_f(p) \cap P(f)$, $\mathrm{index}(q) = \mathrm{index}(p)$.*

Proof. To prove this lemma, we adapt the techniques Lemma 3.5 in [6]. Let U be a locally maximal neighborhood of $H_f(p)$, and let $e > 0$ be a measure cw-expansive constant of f on $H_f(p)$. Then we have $\varepsilon > 0$ such that $B_\varepsilon(H_f(p)) \subset U$. Let $\varepsilon_1 > 0$ be a constant such that $\varepsilon_1 < \min\{e, \varepsilon\}$ and $\sup\{\mathrm{diam} W^{cs}_{\varepsilon_1}(q) : q \in H_f(p)\} < \varepsilon$. For any $q \in H_f(p)$ with $q \sim p$, we let

$$\varepsilon(q) = \sup\{\varepsilon > 0 : f^n(W^{cs}_\varepsilon(q)) \subset W^{cs}_{\varepsilon_1}(f^n(q)) \text{ for all } n \geq 0\}.$$

Let $\varepsilon' = \inf\{\varepsilon(q) : q \in P(f) \text{ with } q \sim p\}$. Then $\varepsilon' > 0$ which means $f^n(W^{cs}_{\varepsilon'}(q)) \subset W^{cs}_{\varepsilon_1}(f^n(q))$ for all $n \geq 0$. Suppose not, we have a sequence $\{q_n\}$ with $q_n \sim p$ such that $\varepsilon(q_n) \to 0$ as $n \to \infty$. Hence we have $0 < m_n < \pi(q_n)$ with $m_n \to \infty$ as $n \to$ and $y_n \in W^{cs}_{\varepsilon(q_n)}(q_n)$ such that $d(f^{m_n}(q_n), f^{m_n}(y_n)) = \varepsilon_1$. Let $I_n = [f^{m_n}(q_n), f^{m_n}(y_n)]$ be an arc joining $f^{m_n}(q_n)$ with $f^{m_n}(y_n) \in W^{cs}_{\varepsilon_1}(f^{m_n}(q_n))$, and let $J_n = f^{-m_n}(I_n)$. Then $J_n \subset W^{cs}_{\varepsilon(q_n)}(q_n)$ and $f^i(J_n) \subset W^{cs}_{\varepsilon_1}(f^i(q_n))$, where $0 \leq i \leq \pi(q_n)$. By Lemma 3.3, we can assume that I_n converges to a closed arc joining

x to y, say, I. Let m_I be the normalized Lebesgue measure on I. We define $\mu \in \mathcal{M}(M)$ by $\mu(C) = m_I(C \cap I)$ for any Borel set C of M. Then $\mu \in \mathcal{M}^*(M)$. Since diam $f^j(I) < \varepsilon_1$ for all $j \in \mathbb{Z}$, we get $0 < \mu(\Gamma^f_{\varepsilon_1}(x) \cap I) \leq \mu(\Gamma_{\varepsilon_1}(x))$. This is contradiction.

Let ε_2 be from Lemma 3.3. For any $y \in W^{cs}_{\varepsilon_2}(q)$ and $z \in W^{cu}_{\varepsilon_2}(q)$, we have $\lim_{n\to\infty} d(f^n(q), f^n(y)) = 0$ and $\lim_{n\to\infty} d(f^{-n}(q), f^{-n}(y)) = 0$.

If $\varepsilon = \min\{\varepsilon', \varepsilon_1\}$, we can see that $f^n(W^{cs}_\varepsilon(x)) \subset W^{cs}_{\varepsilon_1}(f^n(x))$ for $x \in H_f(p)$ and all $n \geq 0$. Moreover, if $y \in W^{cs}_\varepsilon(x) \cap H_f(p)$ then $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$. Consequently we have $W^{cs}_\varepsilon(x) = W^s_\varepsilon(x)$ for any $x \in H_f(p)$. Simiarly we can show that $W^{cu}_\varepsilon(x) = W^u_\varepsilon(x)$ for any $x \in H_f(p)$.

We can take $\delta > 0$ such $W^s_{\varepsilon'}(x) \cap W^u_{\varepsilon'}(y) \neq \emptyset$, whenever $d(x, y) < \delta$ and $x, y \in H_f(p)$. By the λ-lemma, we can see that $W^s_{\varepsilon'}(x) \cap W^u_{\varepsilon'}(y) \subset H_f(p)$. This established that $H_f(p)$ has a local product structure. Since $H_f(p) = \overline{\{q \in P_h(f) : q \sim p\}}$ and $H_f(p)$ has a local product structure, for any periodic point q in $H_f(p)$, we know that $W^s(p) \pitchfork W^u_{\varepsilon'}(q) \neq \emptyset$ and $W^u(p) \pitchfork W^s_{\varepsilon'}(q) \neq \emptyset$. Thus we have index$(q)$ = index (p). □

Proof of Main Theorem To prove that $H_f(p)$ is hyperbolic, it is enough to show that

$$\liminf_{n\to\infty} \|Df^n|_{E(x)}\| = 0 \quad \text{and} \quad \liminf_{n\to\infty} \|Df^{-n}|_{F(x)}\| = 0,$$

for all $x \in H_f(p)$. Suppose $\liminf_{n\to\infty} \|Df^n|_{E(x)}\| \neq 0$ for some $x \in H_f(p)$. For the constant $m \in \mathbb{Z}^+$ taken in Lemma 3.2, let $\psi(x) = \log\|D_x f^m|_{E(x)}\|$. Then we have a sequence $\{j_n\}$ and a f^m-invariant probability measure μ on $H_f(p)$ satisfying

$$\int_{H_f(p)} \psi d\mu = \lim_{n\to\infty} \frac{1}{j_n} \sum_{i=0}^{j_n-1} \log\|D_{f^{mi}(x)} f^m|_{E(f^{mi}(x))}\| \geq 0.$$

By Birkhoff's theorem, together with Mañé's Ergodic Closing Lemma, we can find $q \in \sum_f \cap H_f(p)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\|D_{f^{mi}(q)} f^m|_{E(f^{mi}(q))}\| \geq 0.$$

Here $\sum_f$ is the set of Mañé's Ergodic Closing Lemma. Since q is not a periodi point of f, for $C > 0$ and λ in Lemma 3.2 we can choose

$\lambda < \gamma < 1$ and n_0 such that

$$\frac{1}{n}\sum_{i=0}^{n-1}\log\|D_{f^{mi}(q)}f^m|_{E(f^{mi}(q))}\| \geq \log\gamma$$

when $n > n_0$. By Mañé's ergodic closing lemma we can find $\tilde{f} \in \mathcal{U}_0(f)$ and $\tilde{q} \in \Lambda_{\tilde{f}} \cap P(\tilde{f})$ such that the $\tilde{f}$-orbit of $\tilde{q}$ ε-shadows a part of the f-orbit of g for arbitrarily small $\varepsilon > 0$. Then $\tilde{q}$ is hyperbolic and index($\tilde{q}$)=index(p). We can obtain $g \in \mathcal{V}(\tilde{f}) \subset \mathcal{U}_0(f)$ such that

$$\prod_{i=0}^{k-1}\|D_{g^{im}(\tilde{q})}g^m|_{E(g^{mi}(\tilde{q}))}\| \geq \gamma^k \Rightarrow \prod_{i=0}^{k-1}\|D_{g^{im}(\tilde{q})}g^m|_{E(g^{mi}(\tilde{q}))}\| < C\lambda^k.$$

Observe that we can choose the period $\pi(\tilde{q})$ of $\tilde{q}$ large enough so that $\gamma^k \geq C\lambda^k$, where $k = [\pi(\tilde{q})/m]$. This is a contradiction and hence $\liminf_{n\to\infty}\|Df^n|_{E(x)}\| = 0$ for each $x \in H_p(f)$. Similarly we can show that $\liminf_{n\to\infty}\|Df^{-n}|_{F(x)}\| = 0$ for each $x \in H_f(p)$. □

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